Principle of Symmetric Bracket Invariance as the Origin of First and Second Quantization

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The principle of invariance of the c-number symmetric bracket is used to derive both the quantum operator commutator relation $[\hat{q}, \hat{p}] = i\hbar$ and the time-dependent Schrödinger equation. A c-number dynamical equation is found, which leads to the second quantized field theory of bosons and fermions.

KEY WORDS: quantum theory; fermion; boson; quantum fields.

1. INTRODUCTION

Occasionally in the development of quantum theory and quantum field theory, something fundamental and simple is overlooked. This is the case with the introduction of the ordered Poisson bracket and its consequences. It is shown in this paper that the time-dependent Schrödinger equation and the commutation relation between position and momentum, the quantum bracket $[\hat{q}, \hat{p}] = i\hbar$ (Dirac, 1958, pp. 85–87), is in fact a consequence of the principle of invariance under a one-parameter canonical transformation of the c-number symmetric bracket. Furthermore, the relation between expectation values and classical dynamics, and the probability interpretation of quantum theory are a consequence of this procedure. In addition, a c-number dynamical equation is derived, which provides the fundamental condition for the boson and fermion operator commutation relations.

Although the idea of the symmetric analog of the Poisson bracket has appeared in the theory of differential geometry and algebraic ideals (Droz-Vincent, 1966), and in classical constraint dynamics (Franke and Kálnay, 1970), its clear relevance to fundamental physics has not until now been demonstrated. The idea of the ordered Poisson bracket and related symmetric and antisymmetric brackets

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has been introduced in Garavaglia and Kauffmann (dias-stp-96-17) to provide a c-number analog of the usual boson commutator and fermion anticommutator for quantum fields. From the basic concept of the ordered bracket, the antisymmetric and symmetric brackets are defined. The principle of invariance of the antisymmetric bracket under a one-parameter canonical transformation leads to Hamilton's dynamical equations, and the generator of this transformation is the Hamiltonian. What is new and surprising is that the analogous property for the symmetric bracket leads to Schrödinger's equation, and the generator of the one-parameter canonical transformation in this case is the expectation value of the Hamiltonian operator. Furthermore; these c-number brackets provide a natural derivation of the boson and fermion commutation relations, when operator infinitesimal time development equations are sought, which have the c-number equations as a displacement state expectation value.

In this paper dimensionless phase space coordinates are used such that $q_i \rightarrow q_i/q_o$, $p_i \rightarrow p_i/p_o$, and $q_o p_o = \hbar$. For a given mass M_o, the natural units of length, time, and energy are respectively, $\lambda = 2\pi\hbar/M_o c$, $T_o = 2\pi\hbar/M_o c^2$, and $E_o = M_o c^2 = \hbar\omega_o$. If M_o is chosen to be the Planck mass, $M_P = \sqrt{\hbar c/G_N}$, then these units can be expressed in terms of natural physical constants (Planck's reduced constant $\hbar$, the speed of light c, and the Newtonian gravitational constant G_N).

2. COMPLEX PHASE SPACE AND C-NUMBER BRACKETS

Ordinary classical dynamics is usually discussed in terms of real-valued phase space vector variables of the form $(\vec{q}, \vec{p})$. However, its relation to quantum theory and to fermion systems is much more transparent if one changes these real phase space vector variables to the complex-valued dimensionless phase space vector variables $\vec{a} \equiv (\vec{q} + i\vec{p})/\sqrt{2}$ and their complex conjugates $\vec{a}^* = (\vec{q} - i\vec{p})/\sqrt{2}$. In terms of components of both of these types of phase space vector variables, the usual Poisson bracket of ordinary classical dynamics is

$$\{f, g\} \equiv \sum_k \left(\frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} - \frac{\partial g}{\partial q_k} \frac{\partial f}{\partial p_k} \right)$$

$$= -i \sum_k \left(\frac{\partial f}{\partial a_k} \frac{\partial g}{\partial a_k^*} - \frac{\partial g}{\partial a_k} \frac{\partial f}{\partial a_k^*} \right). \tag{2.1}$$

From the second Poisson bracket representation given above, we abstract the semi-bracket, which we call the ordered Poisson bracket,

$$\{f \circ g\} = \sum_k \frac{\partial f}{\partial a_k} \frac{\partial g}{\partial a_k^*} = \frac{\partial f}{\partial \vec{a}} \cdot \frac{\partial g}{\partial \vec{a}^*}. \tag{2.2}$$

We note that while $\{f \circ g\}$ is linear in each of its two argument functions f and g, it is neither antisymmetric nor symmetric under their interchange. However, it

does satisfy the identity $\{f \circ g\} = \{g^* \circ f^*\}$, which is in algebraic correspondence with the Hermitian conjugation formula for the product of two Hilbert-space operators, that is, $\hat{f}\hat{g} = (\hat{g}^\dagger \hat{f}^\dagger)^\dagger$. From Eq. (2.2) we define the c-number symmetric and antisymmetric brackets

$$\{f, g\}_\pm \equiv \{f \circ g\} \pm \{g \circ f\}, \tag{2.3}$$

where we note $\{f, g\} = -i\{f, g\}_-$. We readily calculate the c-number symmetric and antisymmetric brackets for the components of $\vec{a}$ and $\vec{a}^*$,

$$\{a_i, a_j\}_\pm = 0 = \{a_i^*, a_j^*\}_\pm, \quad \{a_i, a_j^*\}_\pm = \delta_{ij} = \pm\{a_j^*, a_i\}_\pm. \tag{2.4}$$

Infinitesimal canonical transformations, which leave the brackets invariant, are now introduced. The canonical transformations of ordinary classical dynamics are mappings of the complex phase space vectors $\vec{a} \to \vec{A}(\vec{a}, \vec{a}^*)$ and $\vec{a}^* \to \vec{A}^*(\vec{a}, \vec{a}^*)$, which preserve the antisymmetric c-number Poisson bracket relations among the complex phase space vector components. Also we consider the canonical transformations of complex vector phase space mappings, which preserve the c-number symmetric bracket relations among the complex phase space vector components. It is important to note that the complex phase space vectors are related to ordinary classical mechanics phase space coordinates in the case of the antisymmetric bracket; however, in the symmetric bracket case they correspond to the expansion coefficients of either quantum wave functions or the c-number limit of quantum fields.

Specializing now to infinitesimal phase space transformations $\vec{a} \to \vec{A} = \vec{a} + \delta\vec{a}(\vec{a}, \vec{a}^*)$, we readily calculate the c-number antisymmetric and symmetric brackets for the components of $\vec{A}$ and $\vec{A}^*$ to first order in $\delta\vec{a}$ and $\delta\vec{a}^*$,

$$\{A_i, A_j\}_\pm = \frac{\partial(\delta a_j)}{\partial a_i^*} \pm \frac{\partial(\delta a_i)}{\partial a_j^*}, \quad \{A_i^*, A_j^*\}_\pm = \frac{\partial(\delta a_i^*)}{\partial a_j} \pm \frac{\partial(\delta a_j^*)}{\partial a_i}, \tag{2.5}$$

$$\{A_i, A_j^*\}_\pm = \delta_{ij} + \frac{\partial(\delta a_i)}{\partial a_j} + \frac{\partial(\delta a_j^*)}{\partial a_i^*} = \pm\{A_j^*, A_i\}_\pm. \tag{2.6}$$

If we now impose the requirement that this infinitesimal phase space vector transformation preserves the c-number antisymmetric or symmetric bracket relations among the complex phase space vectors, we obtain the three equations,

$$\frac{\partial(\delta a_j)}{\partial a_i^*} = \mp\frac{\partial(\delta a_i)}{\partial a_j^*}, \quad \frac{\partial(\delta a_j^*)}{\partial a_i} = \mp\frac{\partial(\delta a_i^*)}{\partial a_j}, \quad \frac{\partial(\delta a_i)}{\partial a_j} + \frac{\partial(\delta a_j^*)}{\partial a_i^*} = 0. \tag{2.7}$$

The last of these equations is independent of the value of the $\mp$ symbol, and it is satisfied in particular for one-parameter infinitesimal $\delta\vec{a}$ that are of the form

$$\delta a_i = -i(\delta\lambda)\frac{\partial G}{\partial a_i^*}, \quad \delta a_j^* = i(\delta\lambda)\frac{\partial G}{\partial a_j}, \tag{2.8}$$

where $\delta\lambda$ is a real-valued infinitesimal parameter and $G(\vec{a}, \vec{a}^*, \lambda)$ is a real-valued generating function. We thus can readily verify that the last equation in Eq. (2.7) is satisfied. From the two immediately preceding equations, Eq. (2.8), we obtain the form of the equation that governs any continuous one-parameter trajectory of sequential infinitesimal canonical transformations in the complex vector phase space:

$$i\frac{da_i}{d\lambda} = \frac{\partial G}{\partial a_i^*} \quad \text{or} \quad -i\frac{da_i^*}{d\lambda} = \frac{\partial G}{\partial a_i}. \tag{2.9}$$

In the most general circumstance, G may have an explicit dependence on λ. These equations may be rewritten as the pair of real equations:

$$\frac{dq_i}{d\lambda} = \frac{\partial G}{\partial p_i}, \quad \frac{dp_i}{d\lambda} = -\frac{\partial G}{\partial q_i}, \tag{2.10}$$

which are generalized Hamilton's equations.

For the case of ordinary classical dynamics, antisymmetric bracket case (for which $\mp = +$ in Eq. (2.7)), the first two of the group of three equations that were given above are satisfied identically for the one-parameter infinitesimal $\delta\vec{a}$ of the generating function form that has just been given in Eq. (2.8). However, for the symmetric bracket case (for which $\mp = -$ in Eq. (2.7)), the first two of that group of three equations impose the following constraint on those real-valued generating functions $G(\vec{a}, \vec{a}^*, \lambda)$ of continuous one-parameter canonical transformation trajectories:

$$\frac{\partial^2 G}{\partial a_i \partial a_j} = 0 = \frac{\partial^2 G}{\partial a_i^* \partial a_j^*}. \tag{2.11}$$

3. FERMION C-NUMBER DYNAMICS

For the symmetric bracket case, which we call fermion c-number dynamics, the generating functions of the continuous one-parameter trajectories of sequential infinitesimal canonical transformations are constrained to be constant or linear in each of $\vec{a}$ and $\vec{a}^*$, as well as real-valued. The most general form for the generating function is therefore

$$G(\vec{a}, \vec{a}^*, \lambda) = G_0(\lambda) + \sum_k (\tilde{g}_k(\lambda)a_k^* + \tilde{g}_k^*(\lambda)a_k) + \sum_l \sum_m G_{lm}(\lambda)a_l^* a_m, \quad (3.1)$$

where $G_0(\lambda)$ is real and $G_{lm}(\lambda)$ is a Hermitian matrix. Upon putting this constrained form for G into the complex phase space form of the generalized Hamilton's equations, Eq. (2.9), we arrive at

$$i\frac{da_i}{d\lambda} = \tilde{g}_i(\lambda) + \sum_j G_{ij}(\lambda)a_j, \tag{3.2}$$

which is a (possibly) inhomogeneous linear equation of matrix Schrödinger form. If $\tilde{g}_i(\lambda) = 0$, the preceding equation is a general homogeneous type of Schrödinger equation, whereas if $\tilde{g}_i(\lambda) \propto \delta(\lambda - \lambda')$, it is a general propagator type of Schrödinger equation. It is clear that the c-number dynamics of the symmetric bracket case must be linear and describable by a Schrödinger type equation.

The generating functions of the continuous one-parameter canonical transformation trajectories are usually considered to be observables of classical theory when they have no explicit dependence on the parameter. Thus we restrict $G(\vec{a}, \vec{a}^*, \lambda)$ to have no explicit λ-dependence. In the present case it is always possible to suppress the inhomogeneous part if the Hermitian matrix G_{lm} is not singular. This is done by making the canonical transformation

$$a_i \rightarrow A_i = a_i + \sum_j (G^{-1})_{ij}\tilde{g}_j. \tag{3.3}$$

It is easily verified that these transformed A_i also satisfy the c-number symmetric bracket relations. In terms of these A_is, the generalized Hamilton's equations become

$$i\frac{dA_i}{d\lambda} = \sum_j G_{ij}A_j, \tag{3.4}$$

which are of the homogeneous Schrödinger matrix equation form, while G itself becomes

$$G(\vec{A}, \vec{A}^*) = G_0 - \sum_l \sum_m (G^{-1})_{lm}\tilde{g}_l^*\tilde{g}_m + \sum_l \sum_m G_{lm}A_l^*A_m, \tag{3.5}$$

which has no inhomogeneous term.

4. DERIVATION OF THE TIME-DEPENDENT SCHRÖDINGER EQUATION

The result found in Eq. (3.4) following from the invariance of the symmetric bracket can now be used to derive the time-dependent Schrödinger equation. Choosing the parameter λ to be a time parameter t and assuming that the canonical transformation, Eq. (3.3), has been made, the dynamical equation, Eq. (3.4), for a time-independent G_{ij} becomes

$$i\dot{a}_i(t) = \{g(\vec{a}, \vec{a}^*), a_i(t)\}_+ = \sum_j G_{ij}a_j(t). \tag{4.1}$$

Keeping the last term only in Eq. (3.5) and changing $\vec{A} \rightarrow \vec{a}$ and $G(\vec{A}, \vec{A}^*) \rightarrow g(\vec{a}, \vec{a}^*)$, the real-valued generating function becomes

$$g(\vec{a}, \vec{a}^*) = \sum_i \sum_j a_i^*(t)G_{ij}a_j(t). \tag{4.2}$$

The Hermitian matrix element G_{ij} is associated with an Hermitian operator $\hat{G}$ such that

$$G_{ij} = \langle i|\hat{G}|j\rangle,$$

where $|i\rangle$ form a orthonormal complete set of states with identity operator $I = \sum_i |i\rangle\langle i|$. A general state expanded in this basis is

$$|\psi(t)\rangle = \sum_i a_i(t)|i\rangle, \tag{4.3}$$

with $a_i(t) = \langle i \mid \psi(t)\rangle$. From Eq. (4.1), follows the relation

$$\sum_i \dot{a}_i(t)|i\rangle = i\frac{\partial}{\partial t}|\psi(t)\rangle = \sum_i \sum_j |i\rangle G_{ij} a_j(t)$$

$$= \sum_i \sum_j |i\rangle\langle i|\hat{G}|j\rangle a_j(t)$$

$$= \sum_i \sum_j |i\rangle\langle i|\hat{G}|j\rangle\langle j \mid \psi(t)\rangle$$

$$= I\hat{G}|\psi(t)\rangle i\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{G}|\psi(t)\rangle, \tag{4.4}$$

which is the time-dependent Schrödinger equation when $\hat{G}$ is identified with the Hamiltonian operator $\hat{H}(\hat{q}, \hat{p})$.

We see here the difference in the interpretation of the quantities $a_i(t)$ in the case of the antisymmetric and symmetric brackets. In the former these are just the complex coordinates associated with position q_i and momentum p_i; whereas, in the latter, they represent the expansion coefficients of a general quantum state in terms of an orthonormal basis. Both brackets lead to the completeness relation

$$\langle q|\{|\psi(t)\rangle, \langle\psi(t)|\}_\pm|q'\rangle = \delta(q - q'). \tag{4.5}$$

This is seen from

$$\langle q|\{|\psi(t)\rangle, \langle\psi(t)|\}_\pm|q'\rangle = \sum_i \sum_j \langle q|\{a_i, a_j^*\}_\pm|i\rangle\langle j \mid q'\rangle$$

$$= \sum_i \sum_j \langle q|\delta_{ij}|i\rangle\langle j \mid q'\rangle = \langle q|I|q'\rangle = \delta(q - q'). \tag{4.6}$$

5. DERIVATION OF $[\hat{q}, \hat{p}] = i\hbar$

The principle of symmetric bracket invariance leads to quantum mechanics because it leads to the time-dependent Schrödinger equation and to a derivation of

the Dirac bracket relation $[\hat{q}, \hat{p}] = i$. First, one considers the results of the invariance of the antisymmetric bracket under a one-parameter canonical transformation. The time development of a real function $f(\vec{a}, \vec{a}^*, t)$ is given by

$$\dot{f} = -i\{f, H\}_- + \frac{\partial f}{\partial t}, \tag{5.1}$$

and the dynamical equations for the coordinates are

$$\dot{a}_i = -i\{a_i, H\}_- = -i\frac{\partial H}{\partial a_i^*}$$

$$\dot{a}_i^* = -i\{a_i^*, H\}_- = i\frac{\partial H}{\partial a_i}. \tag{5.2}$$

These are equivalent to Hamilton's equations of classical mechanics.

The invariance of the symmetric bracket under a one-parameter canonical transformation gives dynamical equations for the coordinates (wave function expansion coefficients in this case). It is convenient to write Eq. (4.1) and its complex conjugate as

$$i\frac{\partial \vec{a}}{\partial t} = \hat{G} \cdot \vec{a}$$

$$-i\frac{\partial \vec{a}^*}{\partial t} = \vec{a}^* \cdot \hat{G}. \tag{5.3}$$

These are Schrödinger's equations for a_i and a_i^* (i can also be a continuous index). The time development of a real function $\bar{f}(\vec{a}, \vec{a}^*, t) = \vec{a}^* \cdot \hat{F} \cdot \vec{a}$, which depends on the generator for the one-parameter canonical transformation $g(\vec{a}, \vec{a}^*) = \vec{a}^* \cdot \hat{G} \cdot \vec{a}$, is given by

$$\dot{\bar{f}} = \dot{\vec{a}}^* \cdot \hat{F} \cdot \vec{a} + \vec{a}^* \cdot \hat{F} \cdot \dot{\vec{a}} + \vec{a}^* \cdot \frac{\partial \hat{F}}{\partial t} \cdot \vec{a}$$

$$\dot{\bar{f}} = -i\vec{a}^* \cdot [\hat{F}, \hat{G}] \cdot \vec{a} + \overline{\frac{\partial f}{\partial t}}, \tag{5.4}$$

which follows from Eq. (5.3). Here $\hat{F}$ and $\hat{G}$ are Hermitian matrices (operators). For the discrete index case

$$\bar{f}(\vec{a}, \vec{a}^*, t) = \sum_i \sum_j \langle \psi(t) \mid i \rangle \langle i | \hat{F}(t) | j \rangle \langle j \mid \psi(t) \rangle = \langle \psi(t) | \hat{F}(t) | \psi(t) \rangle, \tag{5.5}$$

and

$$\bar{f}(\vec{a}, \vec{a}^*, t) = \int \int \langle \psi(t) \mid p \rangle \langle p | \hat{F}(t) | p' \rangle \langle p' \mid \psi(t) \rangle \, dp \, dp', \tag{5.6}$$

for the continuous index p. The form of $g(\vec{a}, \vec{a}^*)$ shows that classical results are to be associated with expectation values. When $\hat{G}$ is identified with the Hamiltonian operator $\hat{H}$ and $|i\rangle$ is an eigenstate of the Hamiltonian with eigenvalues E_i, the bilinear form of $g(\vec{a}, \vec{a}^*)$ leads to the statistical interpretation of quantum mechanics. This is seen from

$$\bar{H} = \vec{a} \cdot \hat{H} \cdot \vec{a} = \sum_i E_i |a_i|^2 = \langle \psi(t)|\hat{H}|\psi(t)\rangle, \tag{5.7}$$

with

$$\langle \psi \mid \psi \rangle = \sum_i |a_i|^2 = 1.$$

The classical dynamics case (antisymmetric bracket result, Eq. (5.1)) for the Hamiltonian

$$H(a, a^*) = \frac{p^2}{2m} + V(q), \tag{5.8}$$

gives the result for $f(a, a^*) = q$ that $\dot{q} = p/m$. For the c-number symmetric bracket result, Eq. (5.4), to give a result for $\bar{f}$ that corresponds to classical mechanics, one identifies $\hat{G}$ with the Hamiltonian operator $\hat{H}(\hat{p}, \hat{q})$, and observes that $\dot{\bar{q}} = \bar{p}/m$ when $[\hat{q}, \hat{p}] = i$. This is found from the expectation value

$$\bar{f} = \langle \psi(t)|\hat{f}(t)|\psi(t)\rangle = \langle \psi|\hat{U}^\dagger(t)\hat{f}(t)\hat{U}(t)|\psi\rangle, \tag{5.9}$$

with $|\psi\rangle = |\psi(0)\rangle$, $\hat{U}(t) = \exp(-it\hat{H})$ and the relation

$$\frac{d\bar{f}}{dt} = \langle \psi(t)|i[\hat{H}, \hat{f}] + \frac{\partial}{\partial t}\hat{f}(t)|\psi(t)\rangle, \tag{5.10}$$

which corresponds to Eq. (5.4) when Eq. (4.3) is used. Choosing $\hat{f}(t) = \hat{q}$ and using the Hamiltonian operator found from Eq. (5.8), one finds

$$\frac{d\bar{q}}{dt} = \langle \psi(t)|i\left[\frac{\hat{p}^2}{2m}, \hat{q}\right]|\psi(t)\rangle = \frac{\bar{p}}{m}, \tag{5.11}$$

when $[\hat{q}, \hat{p}] = i$, which of course is equivalent to $[\hat{a}, \hat{a}^\dagger] = 1$. The appropriate correspondence between force and the potential function follows from Eq. (5.10) when $\hat{f}(t) = \hat{p}$. This gives the result

$$\frac{d\bar{p}}{dt} = -\langle \psi(t)|\frac{\partial V(q)}{\partial q}|\psi(t)\rangle, \tag{5.12}$$

since the quantum bracket relation between $\hat{p}$ and $\hat{q}$ implies

$$[\hat{p}, V(q)] = -i\frac{\partial V(q)}{\partial q}. \tag{5.13}$$

This is of course the well-known result of Ehrenfest (1927), and the appropriate states to use in the evaluation of these expressions when associating them with corresponding classical equations is the minimum uncertainty displacement states discussed in Section 6. These results clearly shows that quantum mechanics is a consequence of the principle of symmetric bracket invariance, and this is clearly an advance in the understanding of the origin and properties of quantum theory.

The above proof depends upon the association of the quantum operators with observed quantities through the prediction of distributions for the spectrum of the operators and expectation values. Naturally, the generating function $g(\vec{a}^*, \vec{a})$ given in Eq. (4.2), leading to the invariance of the symmetric bracket, is of this form. Since $\hat{G}$ can be identified with the Hamiltonian $\hat{H}(\hat{q}, \hat{p})$, this requires the existence of the expectation values for the operators $\hat{p}$ and $\hat{q}$. A related derivation of the result $[\hat{q}, \hat{p}] = i$, which depends upon the association of distributions and expectation values of the self-adjoint operators $\hat{q}$ and $\hat{p}$ with the classically observed values, is found in Garavaglia (1985), and a similar approach is found in Heslot (1985). The argument of Dirac (1958, pp. 85–87) leading to $[\hat{q}, \hat{p}] = i$ is incorrect because it depends upon the nonclassical concept of noncommuting quantities in the definition of the classical Poisson bracket.

It is easily seen that the argument above for the nonrelativistic Hamiltonian leading to the quantum bracket result applies to the relativistic Dirac Hamiltonian associated with a fermion. Furthermore, the importance of the natural relation between the expectation value of an operator and its observed classical values, which emerges from the principle of invariance of the symmetric bracket, also resolves the dilemma of Dirac where he finds the eigenvalues of $\dot{\hat{q}}$ to be $\pm c$ (Dirac, 1958, pp. 261–263). The correct result for a free relativistic Dirac particle of mass m, momentum p, and energy E is

$$\dot{q} = \frac{p}{E} = \beta, \tag{5.14}$$

where, using $\hbar = c = 1$ and the conventions of Berestetskii et al. (1982),

$$p = \gamma \beta m$$
$$E = \gamma m \tag{5.15}$$
$$\gamma = 1/\sqrt{1 - \beta^2}.$$

This follows from the time derivative of the expectation value

$$\dot{q} = \frac{d}{dt}\langle \psi(t)|\hat{q}|\psi(t)\rangle$$
$$= -i\langle \psi(t)|[\hat{q}, \hat{H}]|\psi(t)\rangle$$
$$= \bar{u}(p)\vec{\gamma}u(p)/2E = \vec{p}/E, \tag{5.16}$$

where for a free Dirac particle

$$\hat{H} = \gamma^0 \vec{\gamma} \cdot \vec{p} + \gamma^0 m,$$

$$\langle x \mid \psi(t) \rangle = \psi(x, t) = \frac{1}{\sqrt{2E}} u(p)\, e^{-ipx}$$

$$px = p^0 t - \vec{p} \cdot \vec{r} \qquad\qquad (5.17)$$

$$\bar{u}(p) = u^\dagger(p)\gamma^0$$

$$\bar{u}(p)u(p) = 2m.$$

This removes the need for the notion of Zitterbewegung, which is associated with the Heisenberg operator but not with the observed mean value of the operator through the expectation value.

6. QUANTUM FIELD OPERATORS ASSOCIATED WITH a_i AND a_i^*

For each index i, one can associate an operator with the complex numbers a_i through the matrix element

$$a_i = \langle a_i | \hat{a}_i | a_i \rangle. \qquad\qquad (6.1)$$

As shown in the next section, the operator relations that are consistent with the infinitesimal time development equations for both bracket relations, Eqs. (4.1) and (5.2), involving the complex numbers a_i are

$$\{a_i, a_j^*\}_\pm = \delta_{ij} = [\hat{a}_i, \hat{a}_j^\dagger]_\pm. \qquad\qquad (6.2)$$

Introducing the notation $a = a_i$, we can discuss both the case of boson operators and fermion operators without loss of generality. In both cases, the states to use in Eq. (6.1) are defined as displacement states

$$|a\rangle = \hat{D}(a)|0\rangle. \qquad\qquad (6.3)$$

In the boson case, the displacement operator is

$$\hat{D}(a) = e^{a\hat{a}^\dagger - a^*\hat{a}}, \qquad\qquad (6.4)$$

such that

$$\sigma(q) = \sigma(p) = 1/\sqrt{2}, \quad \sigma(q)\sigma(p) = 1/2$$

$$\sigma^2(A) = \langle a| \hat{A}^2 |a\rangle - \langle a| \hat{A}|a\rangle^2 \qquad\qquad (6.5)$$

$$a = \langle a|\hat{a}|a\rangle, \quad a^* = \langle a|\hat{a}^\dagger|a\rangle.$$

The interpretation of a_i in this case is clear. The state $|a_i\rangle$ is the minimum uncertainty state, and the a_is are the complex numbers that appear in the antisymmetric bracket, Eq. (2.4), and Hamilton's equations, that is, classical coordinates.

The fermion case can be treated in a similar manner; however, there are some modifications in interpretation. The displacement operator in this case is

$$\hat{D}(\xi) = e^{\xi \hat{a}^\dagger - \xi^* \hat{a}}, \quad \xi = |\xi| \, e^{+i\phi}$$

$$|a\rangle = \hat{D}(\xi) \, e^{-i\phi/2}|0\rangle = \cos(|\xi|) \, e^{-i\phi/2}|0\rangle + e^{+i\phi/2} \sin(|\xi|)|1\rangle. \tag{6.6}$$

This give the following:

$$a = \langle a|\hat{a}|a\rangle = \frac{\sin 2|\xi|}{2} \, e^{i\phi}$$

$$a^* = \langle a|\hat{a}^\dagger|a\rangle = \frac{\sin 2|\xi|}{2} \, e^{-i\phi}$$

$$\langle a|\hat{a}^\dagger \hat{a}|a\rangle = \sin^2 |\xi|, \quad \langle a|\hat{a}\hat{a}^\dagger|a\rangle = \cos^2 |\xi| \tag{6.7}$$

$$\langle a|\hat{a}\hat{a}^\dagger + \hat{a}^\dagger \hat{a}|a\rangle = 1,$$

when

$$\hat{a}|0\rangle = \hat{a}^\dagger|1\rangle = 0$$

$$\hat{a}|1\rangle = |0\rangle, \quad \hat{a}^\dagger|0\rangle = |1\rangle.$$

An analogous calculation for $\sigma(q)$ and $\sigma(p)$ for the fermion case gives

$$\sigma(q) = (1 - \sin^2(2|\xi|) \cos^2(\phi))^{1/2}/\sqrt{2}$$

$$\sigma(p) = (1 - \sin^2(2|\xi|) \sin^2(\phi))^{1/2}/\sqrt{2} \tag{6.8}$$

$$\sigma(q)\sigma(p) \geq 0.$$

The last inequality in Eq. (6.8) does not violate the minimum uncertainty inequality, $\sigma(q)\sigma(p) \geq 1/2$, since $[\hat{q}, \hat{p}] \neq i$, and $\hat{q}$ and $\hat{p}$ are not conjugate coordinates.

7. INFINITESIMAL C-NUMBER TRANSFORMATIONS AND THEIR RELATION TO BOSON AND FERMION OPERATORS

The infinitesimal transformations induced by the c-number symmetric and antisymmetric brackets have analogous relations involving operators, and these lead naturally to the boson and fermion operator relations, Eq. (6.2). For the anti-symmetric bracket, the transformation associated with a time dt is

$$a_i(dt) = a_i(0) + i \, dt\{H(\vec{a}, \vec{a}^*), a_i(0)\}_-, \tag{7.1}$$

and the appropriate operator equation to associate with this c-number equation is

$$\hat{a}_i(dt) = \hat{a}_i(0) + i \, dt[\hat{H}(\hat{a}, \hat{a}^\dagger), \hat{a}_i(0)]. \tag{7.2}$$

With $\hat{a}_i = \hat{a}_i(0)$, the commutation relation for $\hat{a}_i$ and $\hat{a}_j^\dagger$ in this case follows from the relation $[\hat{q}, \hat{p}] = i$, which is a consequence of Eq. (5.11), and is the boson

commutator

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}. \tag{7.3}$$

The associated classical Hamiltonian is found from the normal ordered matrix element

$$H(\vec{a}, \vec{a}^*) = \langle \vec{a} | : \hat{H} : | \vec{a} \rangle, \tag{7.4}$$

with $|\vec{a}\rangle = |a_0\rangle |a_1\rangle \cdots |a_n\rangle$, where $|a_i\rangle$ are the minimum uncertainty states defined in Eq. (6.4). Here normal ordering is defined as moving the operators $\hat{a}_i^\dagger$ to the left according to the boson commutation operation. In this way, the c-number equation, Eq. (7.1), is a consequence of the expectation value of Eq. (7.2), using the displacement states $|\vec{a}\rangle$ found from Eq. (6.4).

The infinitesimal c-number transformation associated with the symmetric bracket implies both the boson commutation and fermion anticommutation relations. For the c-number symmetric bracket, the infinitesimal transformation is

$$a_i(dt) = a_i(0) - i\, dt \{g(\vec{a}, \vec{a}^*), a_i(0)\}_+, \tag{7.5}$$

and the appropriate operator equation to associate with this is

$$\hat{a}_i(dt) = \hat{a}_i(0) + i\, dt [\hat{g}(\hat{a}, \hat{a}^\dagger), a_i(0)], \tag{7.6}$$

with

$$\hat{g} = \vec{\hat{a}}^\dagger \cdot \hat{G} \cdot \vec{\hat{a}}, \tag{7.7}$$

and $\hat{a}_i = \hat{a}_i(0)$. It is now shown that Eq. (7.6) yields the c-number equation, Eq. (7.5), when the operators satisfy either the boson commutator or fermion anticommutator relation, Eq. (6.2). This follows from using

$$
\begin{aligned}
[\hat{g}, \hat{a}_i] &= -\sum_j \sum_k ([\hat{a}_i, \hat{a}_j^\dagger]\hat{a}_k + \hat{a}_j^\dagger [\hat{a}_i, \hat{a}_k]) G_{jk} \\
&= -\sum_j \sum_k (\delta_{ij}\hat{a}_k) G_{jk}, \text{ boson case} \\
&= -\sum_j \sum_k ((1 - 2\hat{a}_j^\dagger \hat{a}_i)\hat{a}_k + 2\hat{a}_j^\dagger \hat{a}_i \hat{a}_k) G_{jk}, \text{ fermion case} \\
&\equiv -\sum_k G_{ik}\hat{a}_k, \tag{7.8}
\end{aligned}
$$

and one finds

$$
\hat{a}_i(dt) = \sum_j (\delta_{ij} - i\, dt\, G_{ij})\hat{a}_j
$$

$$
\langle \vec{a} | \hat{a}_i(dt) | \vec{a} \rangle = \sum_j \langle \vec{a} | (\delta_{ij} - i\, dt\, G_{ij})\hat{a}_j | \vec{a} \rangle \tag{7.9}
$$

$$a_i(dt) = \sum_j (\delta_{ij} - i\,dt\,G_{ij})a_j,$$

which agrees with Eqs. (4.1) and (7.5). Here the state $|\vec{a}\rangle$ is defined as the direct product of displacement states, $|\vec{a}\rangle = |a_0\rangle|a_1\rangle \cdots |a_n\rangle$, found from either Eq. (6.4) for the boson case or Eq. (6.6) for the fermion case.

8. QUANTUM FIELDS

It is seen from the above that the infinitesimal transformations obtained in both the antisymmetric and symmetric bracket case have corresponding operator equations, if the operators $\hat{a}_i$ and $\hat{a}_j^\dagger$ satisfy the boson commutation relations in the former case and the fermion anticommutation relations in the latter. Thus the expansion of quantum fields in these operators is a natural consequence of the relations found for the associated c-numbers. In both cases the usual quantum field expansion (Berestetskii *et al.*, 1982) is

$$\Psi(\vec{r}, t) = \sum_i (\hat{a}_i(t)\psi_i(\vec{r}) + \hat{b}_i^\dagger(t)\psi_i^*(\vec{r})), \tag{8.1}$$

where $\hat{b}_i^\dagger(t) = \hat{a}_i(p_{i0} < 0)$, with four-momentum time component p_0, is an antiparticle creation operator. The associated c-number fields are found by forming the matrix element with the displacement state $|\vec{a}\rangle$ appropriate to either the boson or the fermion case.

The Dirac equation, which is of Schrödinger type, can of course describe a c-number fermion system, but the Klein–Gordon and Maxwell equations, although they are linear, are not of Schrödinger type. For example, in one spatial dimension a discretized version of the Klein–Gordon equation is

$$\ddot{q}_i - (1/(2\Delta x))^2(q_{i+2} - 2q_i + q_{i-2}) + m^2 q_i = 0. \tag{8.2}$$

This can be replaced by the first-order equation pair

$$\dot{q}_i = p_i, \quad \dot{p}_i = (1/(2\Delta x))^2(q_{i+2} - 2q_i + q_{i-2}) - m^2 q_i, \tag{8.3}$$

which is a version of Hamilton's equations for the particular Hamiltonian (time evolution generating function and observable)

$$H(\vec{q}, \vec{p}) = \frac{1}{2}\sum_k \left(p_k^2 + (1/(2\Delta x))^2(q_{k+1} - q_{k-1})^2 + m^2 q_k^2\right). \tag{8.4}$$

The constraint equations, Eq. (2.11), on fermion system c-number generating functions G, which were previously written in terms of the complex $(\vec{a}, \vec{a}^*)$ vector phase space variables, translate in terms of the real $(\vec{q}, \vec{p})$ vector phase space variables

into the pair of real-valued constraint equations:

$$\frac{\partial^2 G}{\partial q_i \partial q_j} = \frac{\partial^2 G}{\partial p_i \partial p_j}, \quad \frac{\partial^2 G}{\partial q_i \partial p_j} = -\frac{\partial^2 G}{\partial q_j \partial p_i}. \tag{8.5}$$

For the discretized Klein–Gordon Hamiltonian given above, we have that

$$\frac{\partial^2 H}{\partial q_i \partial q_{i+2}} = -\left(\frac{1}{2\Delta x}\right)^2 \neq 0 \quad \text{and} \quad \frac{\partial^2 H}{\partial p_i \partial p_{i+2}} = 0, \tag{8.6}$$

which is not in accord with the first of the preceding pair of c-number fermion system generating function constraint equations. Thus the Klein–Gordon equation is not of Schrödinger type and cannot describe a c-number fermion system.

We have seen that c-number fermion dynamics is necessarily described by a Schrödinger type equation, that is, is necessarily already first quantized, and it has no classical version. Therefore, its quantization with anticommutators is inevitably second quantization. On the other hand, the boson commutation relations are consistent with the results of the antisymmetric c-number bracket equations, and the first and second quantized theories involving bosons commutation relations can be directly related to the classical theories through the displacement states, Eq. (6.4).

9. TIME DEVELOPMENT OF a_i AND $\hat{a}_i$

From the infinitesimal transformations that preserve the brackets, it is possible to obtain the global representations of the operators that produce the time development of the coordinates a_i. For the c-number antisymmetric bracket case, the time development operator obtained from Eq. (7.1) is

$$a_i(t) = U(t)a_i = e^{it\delta_-(H)}a_i$$
$$\delta_-(H)a_i = \{H, a_i\}_-$$
$$\delta_-^2(H)a_i = \{H, \{H, a_i\}_-\}_-, \text{ etc.} \tag{9.1}$$
$$a_i(0) = a_i.$$

As an example, if $H = a^*a$, then one finds $a(t) = e^{-it}a$. Under time development, one can show that the antisymmetric bracket is invariant,

$$\{a_i(t), a_j^*(t)\}_- = \{a_i, a_j^*\}_- = \delta_{ij}.$$

The proof is as follows:

$$\{a_i(t), a_j^*(t)\}_- = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(it)^{n+m}}{n!m!}\{\delta_-^n(H)a_i, \delta_-^m(H)a_j^*\}_-$$

$$= \sum_{p=0}^{\infty} \frac{(it)^p}{p!} \sum_{m=0}^{p} \frac{p!}{(p-m)!m!} \{\delta_-^{p-m}(H)a_i, \delta_-^m(H)a_j^*\}_-$$

$$= \sum_{p=0}^{\infty} \frac{(it)^p \delta_-^p(H)}{p!} \{a_i, a_j^*\}_- = e^{it\delta_-(H)}\{a_i, a_j^*\}_- = \delta_{ij}. \quad (9.2)$$

In the above, $p = n + m$, and use has been made of

$$\delta_-^p(H)\{a_i, a_j^*\}_- = \sum_{m=0}^{p} \binom{p}{m} \{\delta_-^{p-m}(H)a_i, \delta_-^m(H)a_j^*\}_-,$$

which follows from the Jacobi identity

$$\delta_-(H)\{a_i, a_j^*\}_- = \{\delta_-(H)a_i, a_j^*\}_- + \{a_i, \delta_-(H)a_j^*\}_-.$$

The time development generated by the c-number symmetric bracket can be studied in a similar manner. Here the time development operator obtained from Eq. (7.5) for the c-number phase space coordinates is

$$a_i(t) = V(t)a_i = e^{-it\delta_+(g)}a_i$$

$$\delta_+(g)a_i = \{g, a_i\}_+$$

$$\delta_+^2(g)a_i = \{g, \{g, a_i\}_+\}_+, \text{ etc.} \quad (9.3)$$

$$a_i(0) = a_i,$$

and $g = \vec{a}^* \cdot \hat{G} \cdot \vec{a}$. Since

$$\{g, a_i\}_+ = \sum_j G_{ij}a_j,$$

one finds

$$a_i(t) = \sum_j \left(\delta_{ij} - itG_{ij} + \frac{(it)^2}{2!} \sum_k G_{ik}G_{kj} + \cdots \right) a_j. \quad (9.4)$$

Defining the operator $\hat{G}$, which must be Hermitian since g is real, as done after Eq. (4.2), we see that Eq. (9.4) becomes

$$a_i(t) = \sum_j \langle i | \left(I - it\hat{G} + \frac{(it)^2}{2!}\hat{G}^2 + \cdots \right) | j \rangle a_j$$

$$= \langle i | e^{-it\hat{G}} | \psi \rangle = \langle i | \psi(t) \rangle, \quad (9.5)$$

since $a_j = \langle j | \psi(0) \rangle = \langle j | \psi \rangle$.

It is now easy to demonstrate that the invariance of the c-number symmetric bracket results from the unitary transformation $\hat{U}(t) = e^{-it\hat{G}}$. This is seen from

$$
\begin{aligned}
\{a_i(t), a_j^*(t)\}_+ &= \{\langle i|\hat{U}(t)|\psi\rangle, \langle\psi|\hat{U}^\dagger(t)|j\rangle\}_+ \\
&= \sum_k \sum_l \{\langle i|\hat{U}(t)|k\rangle\langle k\mid\psi\rangle, \langle\psi\mid l\rangle\langle l|\hat{U}^\dagger(t)|j\rangle\}_+ \\
&= \sum_k \sum_l \langle i|\hat{U}(t)|k\rangle\langle l|\hat{U}^\dagger(t)|j\rangle\{a_k, a_l^*\}_+ \\
&= \sum_k \sum_l \langle i|\hat{U}(t)|k\rangle\langle l|\hat{U}^\dagger(t)|j\rangle\delta_{kl} = \langle i|\hat{U}(t)\hat{U}^\dagger(t)|j\rangle = \delta_{ij}.
\end{aligned}
$$

$$(9.6)$$

In the case of the quantum boson or fermion operators, the invariance of the commutation relations, Eq. (6.2), follows from the unitarity of the time development operator $\hat{U}(t) = e^{-it\hat{H}}$ obtained from Eq. (7.2) for the boson case or $\hat{U}(t) = e^{-it\hat{G}}$ obtained from Eq. (9.5) for the fermion case, such that

$$
\hat{a}_i(t) = \hat{U}^\dagger(t)\hat{a}_i\hat{U}(t) = \sum_{n=0}^{\infty} \frac{(it)^n \hat{D}^n(\hat{H})}{n!}\hat{a}_i
$$

$$
\hat{D}(\hat{H})\hat{a}_i = [\hat{H}, \hat{a}_i]
$$

$$
\hat{D}^2(\hat{H})\hat{a}_i = [\hat{H}, [\hat{H}, \hat{a}_i]], \text{ etc.} \tag{9.7}
$$

$$
\hat{a}_i(0) = \hat{a}_i.
$$

$$
[\hat{a}_i(t), \hat{a}_j^\dagger(t)]_\pm = \hat{U}^\dagger(t)[\hat{a}_i, \hat{a}_j^\dagger]_\pm\hat{U}(t) = \delta_{ij}.
$$

10. ANGULAR MOMENTUM

It is interesting to note that the c-number antisymmetric bracket generates the algebra of orbital angular momentum, and that there is a c-number differential operator representation of the $SU(2)$ Lie algebra. The components of orbital angular momentum are (for i, j, and $k = 1, 2,$ or 3)

$$
l_i = i \sum_i \sum_j \epsilon_{ijk}a_i a_j^*, \tag{10.1}
$$

with ϵ_{ijk} antisymmetric in its indices and $\epsilon_{123} = 1$. For the classical coordinates a_i, the following relations are found:

$$
\{l_i, l_j\}_- = i \sum_k \epsilon_{ijk}l_k,
$$

$$
\{l^2, l_j\}_- = 0
$$

$$\{l_i, q_j\}_- = i \sum_k \epsilon_{ijk} q_k, \quad \{l_i, p_j\}_- = i \sum_k \epsilon_{ijk} p_k. \tag{10.2}$$

A differential operator representation for the $SU(2)$ algebra is given by

$$\hat{J}_\pm = \hat{J}_1 \pm i \hat{J}_2$$

$$\hat{J}_+ = a^* \frac{\partial}{\partial a}, \quad \hat{J}_- = a \frac{\partial}{\partial a^*}$$

$$\hat{J}_3 = \frac{1}{2} \left(a^* \frac{\partial}{\partial a^*} - a \frac{\partial}{\partial a} \right)$$

$$[\hat{J}_3, \hat{J}_\pm] u(j, m) = \pm \hat{J}_\pm u(j, m)$$

$$[\hat{J}_+, \hat{J}_-] u(j, m) = 2 \hat{J}_3 u(j, m) = 2m u(j, m) \tag{10.3}$$

$$u(j, m) = \frac{a^{*j+m} a^{j-m}}{\sqrt{(j+m)!(j-m)!}}$$

$$\hat{J}^2 u(j, m) = \hat{K}(\hat{K} + 1) u(j, m) = j(j + 1) u(j, m)$$

$$\hat{K} = \frac{1}{2} \left(a^* \frac{\partial}{\partial a^*} + a \frac{\partial}{\partial a} \right)$$

$$\hat{J}_\pm u(j, m) = \sqrt{j(j+1) - m(m \pm 1)} \, u(j, m \pm 1).$$

For the functions $u(j, m)$ the inner product is define, with $j \geq j'$, as

$$\langle u(j, m) \mid u(j', m') \rangle = \frac{1}{2\pi N(j, m)} \int_0^{2\pi} \frac{\partial^{4j} u^*(j, m) u(j', m')}{\partial^{2j} a^* \partial^{2j} a} \, d\phi$$

$$= \delta_{jj'} \delta_{mm'} \tag{10.4}$$

$$N(j, m) = \frac{(2j)!(2j)!}{(j+m)!(j-m)!},$$

with $a = \rho \, e^{i\phi}$.

11. CONCLUSIONS

In this paper it has been shown that both quantum theory and the quantum field theory of bosons and fermions are a natural consequence of the principle of invariance of the symmetric bracket, a concept that is analogous to the bracket invariance principle that appears in classical dynamics. Just as the invariance of the c-number antisymmetric bracket under a one-parameter canonical transformation leads to dynamical equations, which determine the classical dynamical flow in coordinate phase space, the invariance of the c-number symmetric bracket under a one-parameter canonical transformation leads to a dynamical equation,

Eq. (7.5), which determines the dynamical flow of quantum states. In the former case, the dynamical equations are Hamilton's equations; however, in the later, the dynamical equation is a time-dependent Schrödinger type equation, Eq. (3.4), which is equivalent to generalized Hamilton's equations, Eq. (2.9) or Eq. (2.10), for the real part q_i and imaginary part p_i of the coordinates $a_i(t) = \langle i \mid \psi(t) \rangle$. The truly remarkable consequences of the principle of invariance of the symmetric bracket are the derivation of the time-dependent Schrödinger equation and the quantum bracket relation $[\hat{q}, \hat{p}] = i\hbar$. This argument makes the time-dependent Schrödinger equation a consequence of bracket invariance, and it replaces with a logical derivation the heuristic conjectures of Schrödinger (1926a,b,c) leading to the discovery of his famous equation. Furthermore, it removes $[\hat{q}, \hat{p}] = i\hbar$ and the time-dependent Schrödinger equation from the status of postulates of quantum theory. Along with these results comes the association naturally of expectation values of quantum operators with corresponding classical quantities, and the statistical interpretation of quantum theory. In addition, the c-number time development equation, Eq. (7.5), found from this principle provides a natural condition for the emergence of the quantum field theory of bosons and fermions, when the antisymmetric bracket is associated with the boson operator commutator and the symmetric bracket is associated with the fermion operator anticommutator. It is clear that both the first quantized and second quantized theories of bosons have an associated c-number dynamics, namely, classical dynamics and classical field theory. These are found from the expectation values and matrix elements of operators using boson minimum uncertainty displacement states. However, fermion c-number dynamics is not classical dynamics, but it is already a first quantized theory, as seen from the derivation in Eq. (4.4). The second quantized version is the quantum field theory of fermions. The c-number coordinates in this case satisfy the c-number dynamical equation, Eq. (7.5), and they are found as matrix elements of fermion operators using the fermion displacement states.

It is clear that the fermion dynamics resulting from the symmetric bracket invariance, allows the gauge couplings that are known to lead to renormalizable theories for fermion dynamics, that is, QED and QCD. The old four-fermion theory of beta decay, which did not require the intermediation of the W boson, clearly has an equation of motion that involves fermion phase space variables in a nonlinear fashion, which thus cannot be of the (necessarily linear) Schrödinger equation type that is here required by invariance of the symmetric bracket. This does not mean that effective theories with nonlinear fermion interactions are not useful approximations. Examples of such approximate theories are the Hubbard model (Fradkin, 1991) and the composite vector boson model (Garavaglia, 1986), where nonlinear interactions may be introduced using path integral methods with auxiliary fields.

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